Note the difference between this and the multiplication of a determinant by a factor. In that case only one row or column contains this factor. From (311) we see that a matrix changes sign only when all its elements change sign; and it vanishes only when all its elements are zero.

Finally, *matrices are added by adding corresponding elements*, thus

$$\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} + \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix} = \begin{Vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{Vmatrix}. \tag{312}$$

This follows from the addition of systems of equations involving the same set of unknowns.

With this brief excursion into matrix algebra we return to our four-terminal network transformations.[1] In matrix form these are

$$\begin{Vmatrix} I_1 \\ I_2 \end{Vmatrix} = \begin{Vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{Vmatrix} \times \begin{Vmatrix} E_1 \\ E_2 \end{Vmatrix}, \tag{275a}$$

$$\begin{Vmatrix} E_1 \\ E_2 \end{Vmatrix} = \begin{Vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ I_2 \end{Vmatrix}, \tag{276a}$$

$$\begin{Vmatrix} I_1 \\ E_2 \end{Vmatrix} = \begin{Vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{Vmatrix} \times \begin{Vmatrix} E_1 \\ I_2 \end{Vmatrix}, \tag{283a}$$

$$\begin{Vmatrix} E_1 \\ I_2 \end{Vmatrix} = \begin{Vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ E_2 \end{Vmatrix}, \tag{284a}$$

$$\begin{Vmatrix} E_1 \\ I_1 \end{Vmatrix} = \begin{Vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{Vmatrix} \times \begin{Vmatrix} E_2 \\ -I_2 \end{Vmatrix}, \tag{285a}$$

$$\begin{Vmatrix} E_2 \\ I_2 \end{Vmatrix} = \begin{Vmatrix} \mathcal{D} & \mathcal{B} \\ \mathcal{C} & \mathcal{A} \end{Vmatrix} \times \begin{Vmatrix} E_1 \\ -I_1 \end{Vmatrix}. \tag{286a}$$

Taken in pairs, these transformations are each others' inverses, and their respective matrices are inverse. The utility of these matrix forms lies primarily in finding the resulting substitutions for several four-terminal networks which are interconnected in various ways. By transforming from one to another, we may also obtain equivalent networks.

[1] The application of matrix algebra to the general treatment of the four-terminal network was first given by F. Strecker and R. Feldtkeller, "Grundlagen der Theorie des allgemeinen Vierpols," E.N.T. 6, pp. 93–112, 1929. See also H. G. Baerwald, "Die Eigenschaften Symmetrischer 4n-Pole . . .", Sitzb. d. Preuss. Akad. d. Wiss. Phys.-Math. Kl. 33, pp. 784–829, 1931.

Considering two four-terminal networks, these may be interconnected in the following five fundamental ways, namely:

(a) cascade
(b) parallel
(c) series
(d) series-parallel
(e) parallel-series.

Fig. 33 illustrates this. In the following we shall treat these fundamental cases only. When more component networks are involved, an extension of the same methods may be applied.

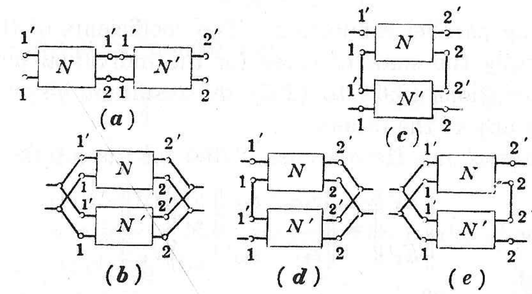

FIG. 33.—Possible interconnections of a pair of dissimilar four-terminal networks.

(a) *Cascade connection.* For the first network in the chain let us assume that the input in terms of the output is given by (285a). For the second we will denote the voltages and currents as well as the coefficients of the matrix by the same letters primed, thus

$$\begin{Vmatrix} E_1' \\ I_1' \end{Vmatrix} = \begin{Vmatrix} \mathcal{A}' & \mathcal{B}' \\ \mathcal{C}' & \mathcal{D}' \end{Vmatrix} \times \begin{Vmatrix} E_2' \\ -I_2' \end{Vmatrix}. \tag{285b}$$

Since the output of the first network equals the input to the second

$$E_2 = E_1'; \; -I_2 = I_1'.$$

Hence, substituting (285b) into (285a), we have

$$\begin{Vmatrix} E_1 \\ I_1 \end{Vmatrix} = \begin{Vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{Vmatrix} \times \begin{Vmatrix} \mathcal{A}' & \mathcal{B}' \\ \mathcal{C}' & \mathcal{D}' \end{Vmatrix} \times \begin{Vmatrix} E_2' \\ -I_2' \end{Vmatrix}, \tag{313}$$

which gives the input in terms of the output for the two networks in cascade. The resultant transformation matrix is obviously obtained by forming the product of the two individual matrices.

(*b*) *Parallel connection.* Here the voltages at the ends of the two networks are common. Using (275*a*) for the top network, and

$$\begin{Vmatrix} I_1' \\ I_2' \end{Vmatrix} = \begin{Vmatrix} y_{11}' & y_{12}' \\ y_{21}' & y_{22}' \end{Vmatrix} \times \begin{Vmatrix} E_1' \\ E_2' \end{Vmatrix} \qquad (275b)$$

for the bottom one, and noting that

$$E_1 = E_1'; E_2 = E_2',$$

we get by adding these matrix equations according to (312)

$$\begin{Vmatrix} I_1 + I_1' \\ I_2 + I_2' \end{Vmatrix} = \begin{Vmatrix} y_{11} + y_{11}' & y_{12} + y_{12}' \\ y_{21} + y_{21}' & y_{22} + y_{22}' \end{Vmatrix} \times \begin{Vmatrix} E_1 \\ E_2 \end{Vmatrix} \qquad (314)$$

for the resulting parallel connection. The coefficients of the resultant matrix are simply the sums of those for the individual matrices. By applying the relations (292) to (295) the resultant y-system may be converted into any of the others.

(*c*) *Series connection.* Here we use (276*a*) for the top network and

$$\begin{Vmatrix} E_1' \\ E_2' \end{Vmatrix} = \begin{Vmatrix} z_{11}' & z_{12}' \\ z_{21}' & z_{22}' \end{Vmatrix} \times \begin{Vmatrix} I_1' \\ I_2' \end{Vmatrix} \qquad (276b)$$

for the bottom one. Then, since

$$I_1 = I_1'; I_2 = I_2',$$

we have by adding (276*a*) and (276*b*)

$$\begin{Vmatrix} E_1 + E_1' \\ E_2 + E_2' \end{Vmatrix} = \begin{Vmatrix} z_{11} + z_{11}' & z_{12} + z_{12}' \\ z_{21} + z_{21}' & z_{22} + z_{22}' \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ I_2 \end{Vmatrix} \qquad (315)$$

for the series connection. Here the resultant transformation matrix is obtained by adding coefficients in the individual z-systems. The resulting z-system may, of course, be converted into any of the others if this should be desired.

(*d*) *Series-parallel connection.* For the treatment of this case we use the transformation (284*a*) because here the input currents and the output voltages are common. Thus if we let (284*a*) be the transformation for one network, and

$$\begin{Vmatrix} E_1' \\ I_2' \end{Vmatrix} = \begin{Vmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{Vmatrix} \times \begin{Vmatrix} I_1' \\ E_2' \end{Vmatrix} \qquad (284b)$$

that for the other, and note that

$$I_1 = I_1'; E_2 = E_2',$$

we have by adding

$$\begin{Vmatrix} E_1 + E_1' \\ I_2 + I_2' \end{Vmatrix} = \begin{Vmatrix} h_{11} + h_{11}' & h_{12} + h_{12}' \\ h_{21} + h_{21}' & h_{22} + h_{22}' \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ E_2 \end{Vmatrix}, \qquad (316)$$

which is the transformation for the series-parallel combination.

(*e*) *Parallel-series connection.* Here (283*a*) together with

$$\begin{Vmatrix} I_1' \\ E_2' \end{Vmatrix} = \begin{Vmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{Vmatrix} \times \begin{Vmatrix} E_1' \\ I_2' \end{Vmatrix} \qquad (283b)$$

are used. Since in this connection

$$E_1 = E_1'; I_2 = I_2',$$

addition of the matrix equations gives

$$\begin{Vmatrix} I_1 + I_1' \\ E_2 + E_2' \end{Vmatrix} = \begin{Vmatrix} g_{11} + g_{11}' & g_{12} + g_{12}' \\ g_{21} + g_{21}' & g_{22} + g_{22}' \end{Vmatrix} \times \begin{Vmatrix} E_1 \\ I_2 \end{Vmatrix} \qquad (317)$$

as the desired relation for the combined system.

Various combinations of these fundamental modes of connection may be treated by applying the same general principles. There is one important restriction to the method, however, which should be pointed out at this time. This has to do with the cases b, c, d, and e. When one four-terminal network operates by itself, it is evident that the current which enters terminal 1′ (Fig. 32) is identical with that which emerges from terminal 1. Both are I_1. Likewise the current I_2 is that which enters 2′ or emerges from 2. When several networks are inter-connected, however, this condition is not assured unless certain other requirements are satisfied. If the current entering 1′ is not the same as that leaving 1, or that entering 2′ is not the same as that leaving 2, the individual transformations which apply for each network by itself no longer hold when the networks are interconnected, and hence the above methods of determining the combined performance fail![1]

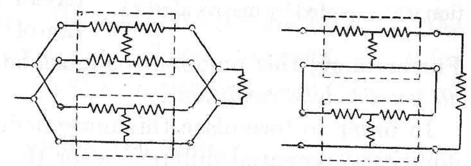

FIG. 34.—Parallel and series inter-connections of four-terminal networks for which the matrix method of determining the composite behavior does not apply.

Fig. 34 illustrates a parallel and a series case for which this condition

[1] H. G. Baerwald, "Der Geltungsbereich der Strecker-Feldtkellerschen Matrizengleichungen von Vierpolsystemen," E.N.T., 9, p. 31, 1932.

exists, i.e., for which the above methods of combination are invalid. The student should convince himself of this by assigning arbitrary finite non-zero resistance values to these networks and calculating the currents entering the terminals of the individual networks for an assumed impressed voltage.

The validity of the above methods of combination may be tested by applying simple rules for the individual cases.[1] These rules are based upon a recognition of the cause of the current unbalance at the pairs of network terminals. Consider the parallel connection of Fig. 33b. Suppose the networks are connected in parallel on their left-hand sides, but that they are individually terminated in such impedances that the voltages E_2 and E_2' are equal. The right-hand sides can then be placed in parallel without disturbing the individual behaviors in any way *provided no potential difference exists between the terminals to be joined,* for, if potential differences do exist between these terminals, then currents will circulate between the networks after the connection is made, and thus the combined behavior will not be given by a superposition of the previous individual behaviors. Furthermore, this requirement must be met for *all frequencies* and for *all possible load conditions.*

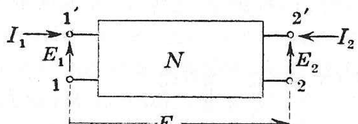

FIG. 35.—Terminal voltage considerations necessary for the determination of the validity conditions for the parallel interconnection when treated by matrix algebra.

In order to formulate this more definitely we must determine these additional potential differences for the connection in question and see under what circumstances they will be zero. In Fig. 35 we have shown, in addition to the usual voltages and currents, the voltage E_{12} which appears between the terminals 1 and 2. By applying the usual principles of lumped network theory, this voltage may be expressed in terms of the voltages E_1 and E_2. Suppose we write for the network N

$$E_{12} = a_1E_1 + a_2E_2, \tag{318}$$

and for the network to be placed in parallel

$$E_{12}' = a_1'E_1' + a_2'E_2'. \tag{318a}$$

Since, for this connection we must have $E_1 = E_1'$ and $E_2 = E_2'$, E_{12} will simultaneously be equal to E_{12}' if

$$a_1 = a_1'; \; a_2 = a_2'. \tag{319}$$

If this condition (319) is fulfilled, then E_{12} will equal E_{12}' for all loads and for all frequencies simultaneously with the condition $E_1 = E_1'$, $E_2 = E_2'$. Then the voltages between the terminals 1' and 2' will also be equal in both networks, and consequently there will be no circulatory currents between them after the parallel connection is made. The relations (319) are, therefore, the necessary and sufficient conditions for the validity of the matrix method of determining the combined behavior for the parallel connection.

[1] O. Brune, E.N.T., Vol. 9, No. 6, p. 234, 1932.

These conditions may now be given a physical interpretation whereby they are more easily applied to a specific case. Suppose the networks are connected in parallel on their input sides but individually short-circuited on their output sides, as shown in Fig. 36a. Then $E_2 = E_2' = 0$, so that the condition $a_1 = a_1'$ for (318) and (318a) becomes $E_{12} = E_{12}'$ which, for the existing connections, is the same as $V = 0$ where V is the voltage appearing between the short-circuited ends. If both networks are reversed, the test for $V = 0$ will correspond to $a_2 = a_2'$. These *two* tests, which for most practical cases may be carried out by inspection, take the place of the conditions (319).

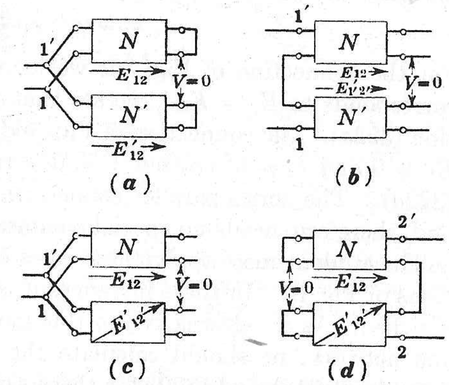

FIG. 36.—Validity tests for matrix methods of determining composite behavior.

For the series connection the validity test is illustrated by Fig. 36b. Here we can write

$$\left.\begin{aligned} E_{12} &= b_1I_1 + b_2I_2 \\ E_{1'2'}' &= b_1'I_1' + b_2'I_2' \end{aligned}\right\}, \tag{320}$$

and since for the series connection it is necessary that

$$I_1 = I_1'; \; I_2 = I_2'$$

shall occur simultaneously with

$$E_{12} = E_{1'2'}',$$

the necessary and sufficient conditions become

$$b_1 = b_1'; \; b_2 = b_2'. \tag{321}$$

The first of these conditions may be checked by having the left-hand sides connected in series while the right-hand sides are open so that $I_2 = I_2' = 0$. Then $V = 0$ corresponds to $b_1 = b_1'$ as shown in Fig. 36b. Repeating the test with the networks reversed is a test for $b_2 = b_2'$.

Finally the tests for the parallel-series connection are shown in Fig.

36c and d. Here the voltages which should be equal for all loads and all frequencies simultaneously with $E_1 = E_1'$ and $I_2 = I_2'$ are E_{12} and $E_{12'}'$. For these we can write

$$\left.\begin{aligned} E_{12} &= c_1E_1 + c_2I_2 \\ E_{12'}' &= c_1'E_1' + c_2'I_2' \end{aligned}\right\}, \tag{322}$$

and thus have for the validity conditions

$$c_1 = c_1'; \; c_2 = c_2'. \tag{322a}$$

For the connection of Fig. 36c we have $I_2 = I_2' = 0$; and since $V = 0$ corresponds to $E_{12} = E_{12'}'$, we see that this is a test for the first condition (322a). The connection of Fig. 36d, on the other hand, makes $E_1 = E_1' = 0$ and $I_2 = I_2'$ so that $V = 0$ is the test for the second condition (322a). The series-parallel connection is essentially the same as this and, therefore, needs no special comment.

The student may apply these rules or tests to the network combinations of Fig. 34. In these instances it is quite obvious that they are not satisfied. As an alternative to this method of testing by inspection of the network, he should calculate the coefficients a_1, a_2, b_1, and b_2 of equations (318) and (320) for these same networks and then apply the criteria (319) and (321). In this way he will appreciate how much simpler it is to apply the tests by inspection than to make the corresponding validity check analytically. When the networks involved are fairly complicated, it may not be possible to apply the tests by inspection. Then the coefficients in the systems (318), (320), and (322) must be determined. This may be quite laborious but it can always be done by the application of usual network principles.

When a larger number of networks are involved, the rules or tests are carried out in the same way. Ends to be paralleled or placed in series are short-circuited or left open respectively, while the opposite ends are connected in the desired manner, and the voltages determined between all terminals which are to be joined. The networks are then reversed and the tests repeated. All these voltages must vanish at all frequencies in order for the matrix method of combination to be valid.

In network synthesis, i.e., in the design of four-terminal networks which are to meet certain prescribed characteristics, we may find that the desired result can be obtained from the combination of several component networks after the matrix fashion. The problem may then be considered solved provided the network combination satisfies the necessary tests for the particular interconnection involved. When these are not satisfied, the difficulty may be overcome in one of several ways.

Cases like those illustrated in Fig. 34 are very simple to handle. In the parallel connection, for example, if we modify the upper network by removing the series resistances from its bottom branches and adding these to the series resistances of its top branches, respectively, it will have the same structure as the lower network, and its external behavior will be unchanged. In this modified form it may be placed in parallel with the other network of similar form without violating the conditions for this connection.

For the series combination of Fig. 34 the situation is still simpler. Here the conditions for the connection are met by inverting the lower network so that the result becomes symmetrical about the horizontal center line.

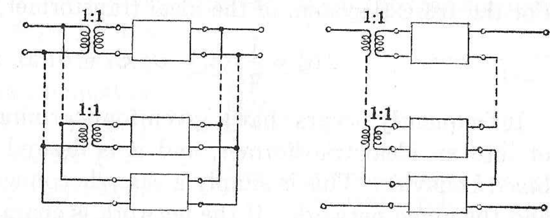

FIG. 37.—Use of ideal transformers in network interconnections for obtaining composite behavior as predicted by matrix methods when validity conditions are not satisfied.

When these or similar measures are insufficient, the situation can always be met by using either input or output transformers of a 1:1 ratio in conjunction with the individual networks. In this way the currents entering and leaving the ends of each network are *forced* to be equal. The transformers must of course be *ideal* so as not to affect the net behavior. This introduces a difficulty from the practical standpoint which can be only approximately met. It should also be pointed out in this connection that transformers need be used only on all but one of the component networks. In two networks, therefore, only one of these needs to be connected through an ideal transformer. Fig. 37 illustrates the general scheme of transformer and network connections for the parallel and the series cases.

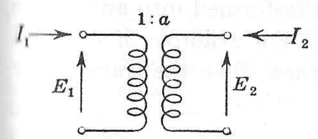

FIG. 38.—Schematic of an ideal transformer considered as a two-terminal pair.

6. Ideal transformers and transformers without loss. Since ideal transformers of various ratios enter into the discussion of four-terminal network analysis quite frequently, it might be well to point out at this time how their characteristics are taken into account.

For the ideal transformer of ratio $1:a$, illustrated in Fig. 38, the following system of equations may be written

$$\left.\begin{aligned} E_1 &= \frac{1}{a}E_2 - 0I_2 \\ I_1 &= 0E_2 - aI_2 \end{aligned}\right| \tag{323}$$